Localized Coherent Structures and their Interactions for the Melnikov Equation

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A general functional separation solution, containing two arbitrary functions, is first obtained for the Melnikov equation by means of the singular manifold method. Some novel localized coherent structures are given by appropriately choosing these arbitrary functions, whose interaction properties are numerically studied. The creation and annihilation phenomenon of dromion structure is reported.

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1. Introduction

The Melnikov equation [1]

$$\mathrm{i}u_t + u_{xx} + uv = 0, \quad v_t + v_y + (|u|^2)_x = 0 \tag{1}$$

is similar to the integrable Zakharov equation in plasma physics to describe the behavior of sonic Langmuir solitons which are Langmuir oscillations trapped in regions of reduced plasma density caused by the ponderomotive force due to a high-frequency field [when $x = y$ in (1)] [2]. The integrability of (1) has been checked by Porsezian [3]. A series of periodic wave solutions in terms of polynomial of Jacobi elliptic functions are also reported [4]. However, (1) possesses many interesting solution structures which have not yet been found. In this paper, using the singular manifold method [5 – 8], we obtain a general functional separation solution of (1) containing two arbitrary functions. By appropriately choosing them, one may get many interesting solution structures. The interaction properties of new types of localized coherent structures are numerically studied. The creation and annihilation phenomenon of dromion structure is reported.

2. General Functional Separation Solution of (1)

To explore the singular structure of (1), we rewrite (1) by $u = q$ and $u^* = r$:

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + qv = 0, \quad -\mathrm{i}r_t + r_{xx} + rv = 0, \\ &v_t + v_y + (qr)_x = 0, \end{aligned} \tag{2}$$

where asterisk stands for the complex conjugation. Truncating the Laurent series of the solution of (2) at the constant level term, one has

$$\begin{aligned} q &= \varphi^{-1} q_0 + q_1, \\ r &= \varphi^{-1} r_0 + r_1, \\ v &= \varphi^{-2} v_0 + \varphi^{-1} v_1 + v_2, \end{aligned} \tag{3}$$

where $\varphi \equiv \varphi(x,y,t)$ is the singular manifold (real function), q_1, r_1 and v_2 are the seed solutions of (2). For simplicity, we take the seed solutions as:

$$q_1 = 0, \quad r_1 = 0, \quad v_2 = v_2(x,\eta), \tag{4}$$

where $v_2(x,\eta)$ with $\eta \equiv y - t$ is an arbitrary function of indicated variables. Substituting (3) and (4) into (2) and equating the coefficients of like powers of φ to zero yields

$$\begin{aligned} &2\varphi_x^2 q_0 + q_0 v_0 = 0, \quad 2\varphi_x^2 r_0 + r_0 v_0 = 0, \\ &\varphi_t v_0 + \varphi_y v_0 + \varphi_x q_0 r_0 = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &-\mathrm{i}\varphi_t q_0 - \varphi_{xx} q_0 - 2\varphi_x q_{0x} + q_0 v_1 = 0, \\ &\mathrm{i}\varphi_t r_0 - \varphi_{xx} r_0 - 2\varphi_x r_{0x} + r_0 v_1 = 0, \\ &v_{0t} - \varphi_t v_1 - \varphi_y v_1 + v_{0y} + (q_0 r_0)_x = 0, \end{aligned} \tag{6}$$

$$\begin{aligned} &\mathrm{i}q_{0t} + q_{0xx} + q_0 v_2 = 0, \\ &-\mathrm{i}r_{0t} + r_{0xx} + r_0 v_2 = 0, \\ &v_{1t} + v_{1y} = 0. \end{aligned} \tag{7}$$

It follows from (5) that

$$v_0 = -2\varphi_x^2, \quad q_0 r_0 = 2\varphi_x(\varphi_t + \varphi_y). \tag{8}$$

From (6), using (8), one gets

$$v_1 = -\frac{2\varphi_x(\varphi_{xt}+\varphi_{xy})}{\varphi_t+\varphi_y} + 2\varphi_{xx}, \quad q_0 = F(y,t)\exp\left[\frac{1}{2}\int\left(-\mathrm{i}\frac{\varphi_t}{\varphi_x} + \frac{\varphi_{xx}}{\varphi_x} - \frac{2(\varphi_{xt}+\varphi_{xy})}{\varphi_t+\varphi_y}\right)\mathrm{d}x\right],$$
$$r_0 = G(y,t)\exp\left[\frac{1}{2}\int\left(\mathrm{i}\frac{\varphi_t}{\varphi_x} + \frac{\varphi_{xx}}{\varphi_x} - \frac{2(\varphi_{xt}+\varphi_{xy})}{\varphi_t+\varphi_y}\right)\mathrm{d}x\right], \tag{9}$$

where F and G are the functions of integration. The substitution of the first equation of (9) into the last of (7) yields the singular manifold equation

$$\begin{aligned}&(\varphi_t+\varphi_y)[(\varphi_{xt}+\varphi_{xy})^2 - (\varphi_{xxt}+\varphi_{xxy})(\varphi_t+\varphi_y)] \\ &+\varphi_x[(\varphi_{xtt}+2\varphi_{xyt}+\varphi_{xyy})(\varphi_t+\varphi_y) \\ &-(\varphi_{xt}+\varphi_{xy})(\varphi_{tt}+2\varphi_{yt}+\varphi_{yy})] = 0,\end{aligned} \tag{10}$$

which has a special general solution

$$\varphi = f(x, y-t) + g(y,t), \tag{11}$$

where f and g are arbitrary functions of indicated variables. Substituting the last two equations of (9) into the first two of (7), we obtain

$$F_t = G_t = g_t = 0,$$
$$v_2 = -\frac{1}{2}\int\frac{f_{\eta\eta}f_x - f_\eta f_{x\eta}}{f_x^2}\mathrm{d}x + \frac{2f_{xxx}f_x - f_{xx}^2 - f_\eta^2}{4f_x^2}. \tag{12}$$

It follows from the relation of q_0 and r_0, i. e. (8) and (9), that $FG = 2f_xg_y/|f_x|$. Thus, we obtain a general functional separation solution of (2):

$$q = \frac{F}{f+g}\exp\left[\frac{1}{2}\int\left(\mathrm{i}\frac{f_\eta}{f_x} + \frac{f_{xx}}{f_x}\right)\mathrm{d}x\right],$$
$$r = \frac{G}{f+g}\exp\left[\frac{1}{2}\int\left(-\mathrm{i}\frac{f_\eta}{f_x} + \frac{f_{xx}}{f_x}\right)\mathrm{d}x\right], \tag{13}$$
$$v = -\frac{2f_x^2}{(f+g)^2} + \frac{2f_{xx}}{f+g} + v_2,$$

where $F \equiv F(y)$, $G \equiv G(y)$, $g \equiv g(y)$ and $f \equiv f(x,\eta)$ with $\eta = y-t$ are arbitrary functions of indicated variables with $FG = 2f_xg_y/|f_x|$, and v_2 is given by the last equation of (12).

3. Novel Solution Structures of (1)

Taking into account our notation in (2), i. e. $q = u$ and $r = u^*$, we have $q = r^*$ as far as (1) is concerned. Using this condition in (13), we obtain

$$|F|^2 = 2f_xg_y/|f_x|. \tag{14}$$

Therefore, from the results of the previous section, we find that the most important physical quantity $|u|^2$ of the original equation (1) takes the form

$$|u|^2 = \frac{2f_xg_y}{(f+g)^2}. \tag{15}$$

It is worth to note that although the functions $f(x, y-t)$ and $g(y)$ in formula (15) are arbitrary, $f_xg_y > 0$ is owing to the condition (14). It is due to the arbitrariness of functions f and g that one may obtain a diversity of solution structures by appropriately choosing them in (15). Several interesting cases are considered as examples in what follows.

Case 1. $f = \exp[\tanh(kx) + l_1(y-t)]$, $g = \exp[\tanh(l_2y)] + A$.

In this case, from (15), we have solution

$$\begin{aligned}|u|^2 = {}& 2kl_2\exp[\tanh(kx)+\tanh(l_2y)+l_1(y-t)] \\ &\cdot \mathrm{sech}^2(kx)\mathrm{sech}^2(l_2y) \\ &\cdot (\exp[\tanh(kx)+l_1(y-t)] \\ &\quad + \exp[\tanh(l_2y)]+A)^{-2},\end{aligned} \tag{16}$$

where k, l_1, l_2, and A are arbitrary constants, which demand $kl_2 > 0$. (16) is a dromion solution. The evolution process is depicted in Fig. 1 with the parameters $k = 1$, $l_1 = 1$, $l_2 = 1$, $A = 1$ and $t = -5, -2, 0, 2, 5$, respectively. From the figures, one can easily see the process of creation and annihilation of the dromion.

Case 2. $f = \tanh[k_1x + l_1(y-t)] + \tanh[k_2x + l_2(y-t)] \equiv \tanh\xi_1 + \tanh\xi_2$, $g = \tanh(ly) + A$.

It follows from (15) that

$$|u|^2 = \frac{2l(k_1\mathrm{sech}^2\xi_1 + k_2\mathrm{sech}^2\xi_2)\mathrm{sech}^2(ly)}{[\tanh\xi_1 + \tanh\xi_2 + \tanh(ly) + A]^2}, \tag{17}$$

where $k_i > 0$, $l > 0$, l_i and $A > 3$ are arbitrary constants. These statements are valid for all the following equations, unless otherwise explained. (17) is a two-dromion-like structure. The interaction is nonelastic, and its details are shown in Fig. 2, with the parameter values $k_1 = 1$, $k_2 = 2$, $l_1 = 1$, $l_2 = -1$, $l = 1$, $A = 4$, and $t = -5, 0, 5$, respectively.

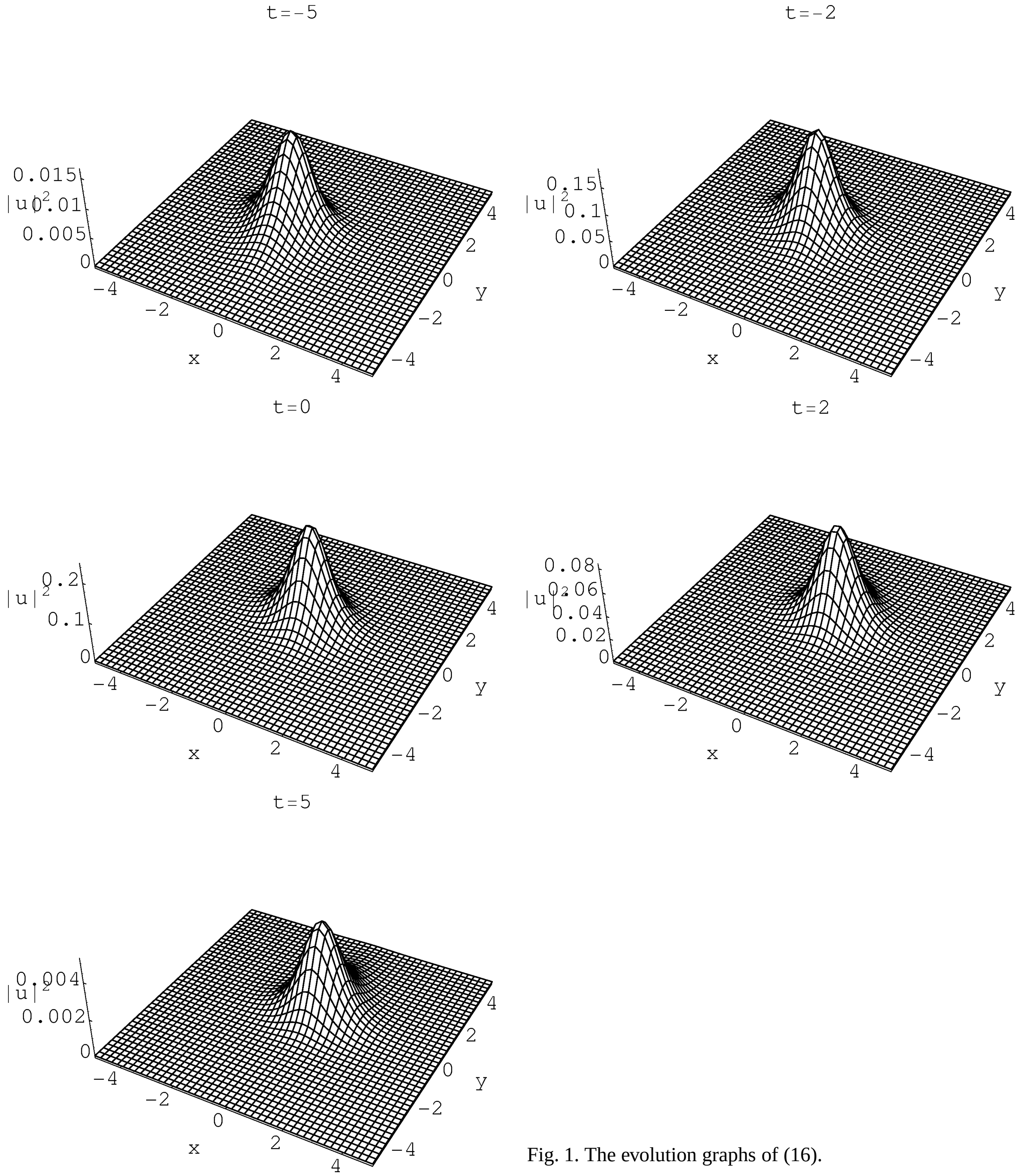

Fig. 1. The evolution graphs of (16).

Case 3. $f = \tanh(\tanh\xi_1) + \tanh(\tanh\xi_2) \equiv \tanh\eta_1 + \tanh\eta_2$, $g = \tanh[\tanh(ly)] + A \equiv \tanh\eta + A$. The direct calculation, from (15), yields

$$|u|^2 = \frac{2l[k_1 \mathrm{sech}^2\eta_1 \mathrm{sech}^2\xi_1 + k_2 \mathrm{sech}^2\eta_2 \mathrm{sech}^2\xi_2]\mathrm{sech}^2\eta\, \mathrm{sech}^2(ly)}{[\tanh\eta_1 + \tanh\eta_2 + \tanh\eta + A]^2}, \qquad (18)$$

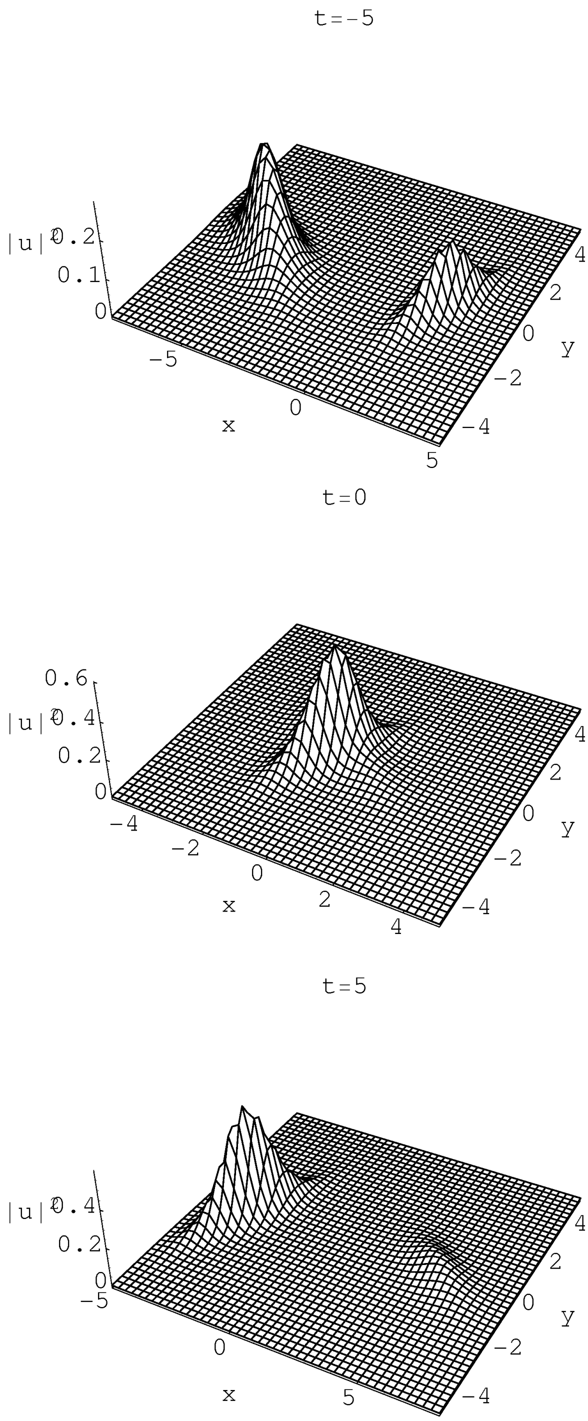

Fig. 2. The evolution graphs of (17).

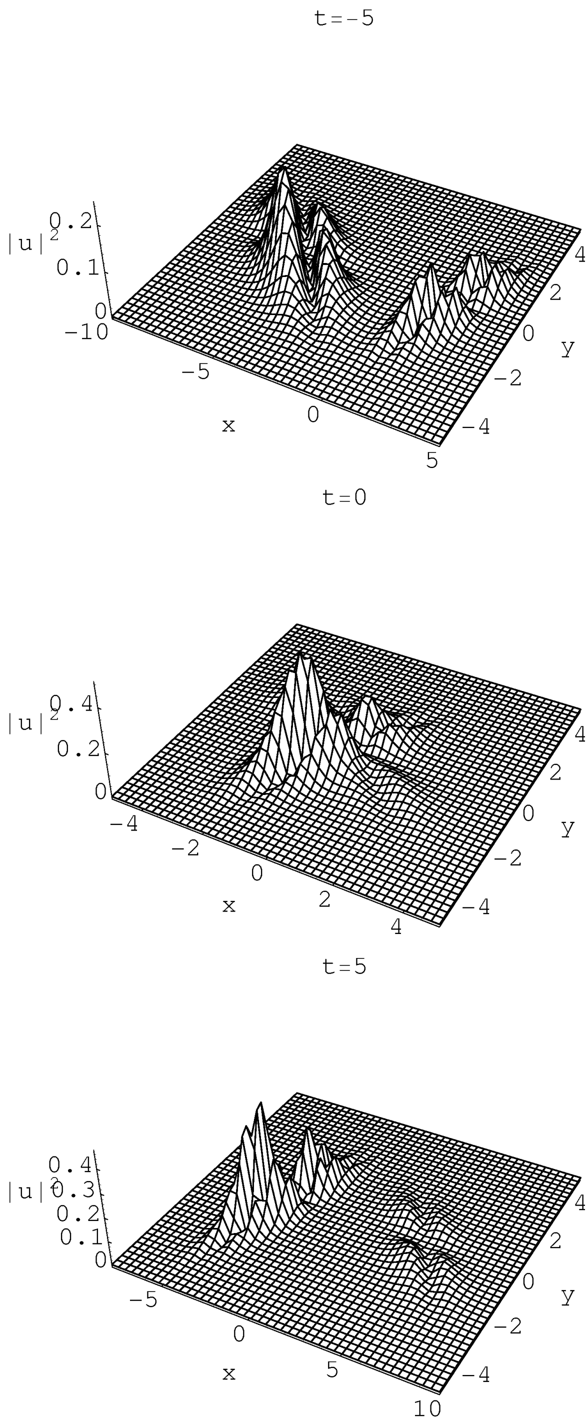

Fig. 3. The evolution graphs of (21).

which is also a two-dromion-like structure. Its evolution graphs are similar to those in Fig. 2 and thus omitted.

Case 4. $f = \exp(\eta_1) + \exp(\eta_2),\ g = \exp(\eta) + A.$

The formula (15) gives another two-dromion-like solution

$$|u|^2 = \frac{2l[k_1\exp(\eta_1)\mathrm{sech}^2\xi_1 + k_2\exp(\eta_2)\mathrm{sech}^2\xi_2]\exp(\eta)\mathrm{sech}^2(ly)}{[\exp(\eta_1) + \exp(\eta_2) + \exp(\eta) + A]^2}. \tag{19}$$

The interaction is also nonelastic, and the graphs of evolution are similar to Fig. 2.

Case 5. $f = \arctan\eta_1 + \arctan\eta_2,\ g = \arctan\eta + A.$

It follows from (15) that

$$|u|^2 = \frac{2l[k_1(1+\eta_1^2)^{-1}\mathrm{sech}^2\xi_1 + k_2(1+\eta_2^2)^{-1}\mathrm{sech}^2\xi_2]\mathrm{sech}^2(ly)}{(1+\eta^2)[\arctan\eta_1 + \arctan\eta_2 + \arctan\eta + A]^2}, \tag{20}$$

which is still a two-dromion-like structure with the same interaction property as (17).

Case 6. $f = \tanh^3\xi_1 + \tanh^3\xi_2,\ g = \tanh^3(ly) + A.$

In this case, we have the multi-solitary wave excitation

$$|u|^2 = \frac{18l(k_1\tanh^2\xi_1\mathrm{sech}^2\xi_1 + k_2\tanh^2\xi_2\mathrm{sech}^2\xi_2)\tanh^2(ly)\mathrm{sech}^2(ly)}{[\tanh^3\xi_1 + \tanh^3\xi_2 + \tanh^3(ly) + A]^2}, \tag{21}$$

whose interaction property is illustrated in Fig. 3 with the same parameters as those of Fig. 2. It is found that the interaction of two group solitary waves is nonelastic.

4. Conclusion and Discussion

Using the singular manifold method, we obtain a general functional separation solution, which contains two lower-dimensional arbitrary functions, for the Melnikov equation. By choosing appropriately these arbitrary functions, some novel solutions to the equation of interest are given. The interaction properties of them are numerically studied, and it is found that they are all nonelastic. Due to the arbitrariness of functions f and g in (15), one may obtain a diversity of novel structures of the solution for (1). Six cases are considered in this paper and they are new types of structures which are not reported previously in the literature to the best of knowledge. And many interesting other cases have to be left to readers due to the limitation of space.

The singular manifold method is a powerful tool for obtaining exact solutions of nonlinear partial differential equations (PDEs). We think that the singular manifold equation, for example (10) in this paper, may contain a great deal of information about the PDE. It is worth studying further if (10) has other types of solutions beyond (11).

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